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INDUCTIVE DEFINITIONS, MODELS OF COMPREHENSION
AND INVARIANT DEFINABILITY

Prepublication

Inductive definitions, models of comprehension and invariant definability [*]

by

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ABSTRACT

The connections between inductive definability and models of comprehension are studied.

Let $U = <A, R_1, \ldots, R_n>$ be an infinite structure and let I_ϕ be a set inductively defined by a formula ϕ of the second order language L_2^U.

We prove in §3 that if A is a model of Δ_1^1-Comprehension relativized to ϕ and ϕ is A-absolute then for every η smaller than the height of A ($h(A)$) I_ϕ^η is in A. If A is a β-structure which satisfies $\sum_1^1$-Comprehension relativized to ϕ and WF(X) and ϕ is A-absolute then I_ϕ is in A and $\|\phi\| < h(A)$. These results imply that Barwise-Grilliot theorem is false in the case of uncoutable acceptable structures.

In §4 we study the notion of invariant definability over models of Δ_1^1-Comprehension. We prove that if U is a countable, acceptable structure and T an inductive theory in L_2^U which extends Δ_1^1-Comprehension and has a model over U then the hyperelementary relations of U are exactly the sets invariantly definable over all models of U. We show that this theorem is false in the case of uncountable acceptable structures. We prove that if U is an acceptable structure then for some inductive set I every set invariantly definable over models of Δ_1^1-Comprehension is hyperelementary in I.

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[*]
This paper is not for review; it is meant for publication elsewhere.

§1. INTRODUCTION

This paper is devoted to the study of connections between inductive definability and models of comprehension. The basic question to which we want to find an answer is the following.

Let U be an infinite structure. Suppose that I_ϕ is a set inductively defined by a second order formula ϕ. Which properties of a second order structure A over U imply that I_ϕ or some of its stages I_ϕ^η belong to $|A|$?

We prove in §3 that if A satisfies Δ_1^1-Comprehension scheme relativized to ϕ and ϕ is A-absolute than for every η smaller than the height of A $(h(A))$ $I_\phi^\eta \in |A|$.

If we assume that A is a β-structure for which ϕ is A-absolute and A satisfies $\sum_1^1$-Comprehension relativized to ϕ and WF(X) then $\|\phi\| < h(A)$ and $I_\phi \in |A|$.

Using these results we prove that Barwise-Grilliot theorem (see MOSCHOVAKIS [2] p. 140) is false in the case of uncountable acceptable structures.

The basic problem to which we are unable to find the answer is the following: is it true that hyperelementary relations on an arbitrary infinite structure U form a model (the smallest one) of Δ_1^1-Comprehension?

In §4 we study the notion of invariant definability over models of Δ_1^1-Comprehension. We prove that if U is a countable acceptable structure and T is an inductive theory in L_2^U which extends Δ_1^1-Comprehension and has a model over U then the hyperelementary relations of U are exactly the sets invariantly definable over models of T. This theorem is not true in the case of uncountable acceptable structures.

Finally we prove that if U is acceptable then for some inductive set I every set invariantly definable over models of Δ_1^1-Comprehension is hyperelementary in I.

Unfortuanately we are unable to give the exact characterization of sets which are invariantly definable over all models of Δ_1^1-Comprehension in the case of an arbitrary infinite structure U.

We thank Mr. E. Alward and prof. G. Kreisel for helpful remarks about Δ_1^1-Comprehension.

§2. PRELIMINARIES

Throughout the paper the letters η,ξ,τ,σ always denote ordinals. If A is a set we use small latin letters to denote the elements of A and capital letters to denote relations on A of any (finite) number of arguments. By A^n $(n \geq 1)$ we denote the set of all n-tuples of elements of A. If X is a set $P(X)$ denotes its power set. $\bar{x}$ denotes a sequence $x_1,\ldots,x_n$ of elements of A and $\bar{X}$ denotes a sequence $X_1,\ldots,X_n$ of relations on A.

For the convenience of the reader we recall here some definitions and notations which can be found in MOSCHOVAKIS [2] and MOSCHOVAKIS [3].

The first order language over a set A, L^A has an infinite list of individual variables $x,y,z,\ldots$, a constant b for each element b of A, an infinite list $X,Y,Z,\ldots$ of n-ary relation variables for each $n \geq 1$ and a constant P for each relation P on A. In forming formulas of L^A the quantifiers $\exists$ and $\forall$ are applied only to individual variables.

The second order language over A, L_2^A is obtained by allowing quantification of the relation variables in the language L^A. For convenience we assume that the formulas of the type $X = Y$ are not well formed formulas (we may write instead $\forall x(X(x) \leftrightarrow Y(x)))$.

Let $U = \langle A, R_1, \ldots, R_n \rangle$ be a structure (that is to say $R_1, \ldots, R_n$ are relations over A). The first (second) order language $L^U(L_2^U)$ for U consists of those formulas of the language $L^A(L_2^A)$ whose relation constants are among $=, R_1, \ldots, R_n$. We sometimes write $\bar{x} \in X$ instead of $X(\bar{x})$.

Formulas of L^U are called arithmetical formulas. All structures considered are infinite.

A is called a second order structure over $U = \langle A, R_1, \ldots, R_n \rangle$ if $A = \langle A, |A|, R_1, \ldots, R_n \rangle$ where $|A| \subset \bigcup_{n \geq 1} P(A^n)$. While interpreting a formula ϕ of L_2^U on A we assume that the second order quantifiers of ϕ range over $|A|$. For simplicity $P(U)$ denotes the second order structure $\langle A, \bigcup_{n \geq 1} P(A^n), R_1, \ldots, R_n \rangle$.

We sometimes write $\phi(\bar{x},\bar{X})$ instead of $P(U) \models \phi(\bar{x},\bar{X})$.

DEFINITION 1. Let A be a second order structure over U.

A formula $\phi(x_1,\ldots,x_k,X_1,\ldots,X_n)$ of L_2^U with free variables indicated is called A-absolute if

$$A \models \phi \, [x_1,\ldots,x_k,X_1,\ldots,X_n] \Leftrightarrow P(U) \models \phi \, [x_1,\ldots,x_n,X_1,\ldots,X_n]$$

for all $x_1,\ldots,x_k \in A$ and $X_1,\ldots,X_n \in |A|$.

WF(X) is the following formula of L_2^U where $U = \langle A,R_1,\ldots,R_n \rangle$ WF(X) $\leftrightarrow$ X is a well-founded transitive relation on A. i.e.

$$WF(X) \leftrightarrow \forall \, x,y,z \, \left[\Big((x,y) \in X \wedge (y,z) \in X\Big) \to (x,z) \in X\right] \wedge$$

$$\wedge \, \forall \, S \, \left[\exists x(x{\in}S) \to \exists y(y{\in}S \wedge \forall x(x{\in}S \to (x,y){\notin}X))\right]$$

DEFINITION 2. A second order structure A over U is called a β-structure if the formula WF(X) is A-absolute.

If $\phi(x_1,\ldots,x_k,X_1,\ldots,X_n)$ is a formula of L_2^U with free variables indicated we say that ϕ is a $(k,r_1,\ldots,r_n)$-formula to indicate the fact that ϕ has exactly k free individual variables and for every $i \le n$ free variable X_i ranges over r_i-ary relations.

DEFINITION 3. Let $\phi(\bar{x},X,\bar{X})$ be a $(m,\ell,r_1,\ldots,r_n)$-formula of L_2^U and $\Psi(\bar{y},\bar{y}_1,\bar{Y})$ be a $(\ell+p,p_1,\ldots,p_r)$-formula of L_2^U. Suppose that $\phi(\bar{x},X,\bar{X})$ and $\Psi(\bar{y},\bar{y}_1,\bar{Y})$ have no variables in common. Then $\phi_{\bar{y}}(\bar{x},\Psi(\bar{y},\bar{y}_1,\bar{Y}),\bar{X})$ is a $(m+p,r_1,\ldots,r_n,p_1,\ldots,p_r)$-formula of L_2^U obtained from $\phi(\bar{x},X,\bar{X})$ by replacing all the atomic formulas of the form $\bar{z} \in X$ occuring in $\phi(\bar{x},X,\bar{X})$ by $\Psi(\bar{z},\bar{y}_1,\bar{Y})$.

LEMMA 1. *Suppose that $\phi(\bar{x},X,\bar{X})$ and $\Psi(\bar{y},\bar{y}_1,\bar{Y})$ are formulas of L_2^U which satisfy the above conditions. Then for all $\bar{x}$, $\bar{y}_1$, $\bar{X}$ and $\bar{Y}$*

$$P(U) \models \phi_{\bar{y}}(\bar{x},\Psi(\bar{y},\bar{y}_1,Y),\bar{X}) \Leftrightarrow P(U) \models \phi(\bar{x},A,\bar{X})$$

where

$$A = \{\bar{y} \, : \, P(U) \models \Psi(\bar{y},\bar{y}_1,\bar{Y})\}$$

PROOF. By induction on the length of the formula $\phi(\bar{x},X,\bar{X},)$. $\square$

DEFINITION 4. Suppose that $X \subset A^2$ is a binary relation on A. Let

$$Fld(X) = \{x : \exists y((x,y) \in X \vee (y,x) \in X)\}.$$

If $z \in Fld(X)$ then

$$X|_z = \{(x,y) : (x,y) \in X \wedge (y,z) \in X\}.$$

By transfinite induction we define the classes of wellfounded transitive relations on A

$$WF(\sigma) = \{X : X \text{ is a wellfounded transitive relation on A and}$$

$$\forall z(z \in Fld(X) \rightarrow X\upharpoonright_z \in WF(\tau) \text{ for some } \tau < \sigma\}$$

It is easy to see then that

$$WF(X) \leftrightarrow X \in WF(\sigma) \text{ for some } \sigma.$$

Suppose that $WF(X)$. By $\|X\|$ we denote the least σ such that $X \in WF(\sigma)$. If $x \in Fld(X)$ then $\|X\upharpoonright_x\| < \|X\|$.
If $(x,y) \in X$ then $X\upharpoonright_x = X\upharpoonright_y\upharpoonright_x$. If $\sigma < \|X\|$ then for some $x \in Fld(X)$ $\sigma = \|X\upharpoonright_x\|$.
   If X and Y are two binary relations on A then by $X \leq Y$ we mean the following formula of L_2^A.

$$\exists Z(Z \text{ is a 1-1 function from } Fld(X) \text{ into } Fld(Y)$$
$$\text{and } \forall x,y ((x,y) \in X \rightarrow (Z(x),Z(y)) \in Y))$$

We define then

$$X < Y \leftrightarrow \exists z (z \in Fld(Y) \wedge X \leq Y\upharpoonright_z)$$

It is easy to see then that if $\overline{\text{WF}}(Y)$ then

$$X \leq Y \Rightarrow \|X\| \leq \|Y\|$$

$$X < Y \Rightarrow \|X\| < \|Y\|$$

DEFINITION 5. Let A be a second order structure over U. We define

$$h(A) = \sup(\|X\|+1 : P(U) \models \text{WF}(X) \text{ and } X \in |A|)$$

We call $h(A)$ the height of A.

DEFINITION 6. Let K be a class of structures over $U = < A, R_1, \ldots, R_n >$. Then

$$\cap K = \{X : X \in |A| \text{ for all } A \in K\}$$

$$\text{Def } (K) = \{X : X \in \underset{n \geq 1}{U} P(A^n) \text{ and for some formula } \phi(\overline{x}) \text{ of } L_2^U$$
$$\text{with free variables indicated which is } A\text{-absolute}$$
$$\text{for all } A \in K$$
$$P(U) \models \forall \overline{x} \ (\overline{x} \in X \leftrightarrow \phi(\overline{x}))\}$$

Thus $\cap K$ is the intersection of all the structures belonging to K whereas Def (K) is the collection of all sets invariantly definable over K.

If T is a set of sentences of L_2^U (i.e. a theory in L_2^U) then

Mod $(T) = \{A : A \text{ is a second order structure over } U \text{ and } A \models T\}$.

DEFINITION 7. Let ϕ_0 be a formula of L_2^U. $\text{Ar}(\phi_0)$ is the smallest class of formulas of L_2^U such that

i) $\phi_0 \in \text{Ar}(\phi_0)$

ii) every arithmetical formula of L_2^U is in $\text{Ar}(\phi_0)$

iii) if $\phi, \Psi \in \text{Ar}(\phi_0)$ then $\neg \phi, \phi \lor \Psi, \exists x \phi \in \text{Ar}(\phi_0)$.

A formula Ψ of L_2^U is a $\sum_1^1(\phi_0)$ formula if it is of the form $\exists X_1 \ldots \exists X_n \ \phi$ for some formula $\phi \in \text{Ar}(\phi_0)$.

We call a relation $R(\bar{x},\bar{X})$ ($R \subset A^n \times P(A^{n_1}) \times \ldots \times P(A^{n_k})$ for some $n, n_1, \ldots, n_K$) arithmetical(Σ_1^1) if for some arithmetical (Σ_1^1) formula ϕ of L_2^{U}

$$R = \{(\bar{x},\bar{X}) : P(U) \models \phi(\bar{x},\bar{X})\}$$

A relation R is Π_1^1 if $\neg R$ is Σ_1^1 and is Δ_1^1 is both R and $\neg R$ are Σ_1^1.
Recall that for $n \geq 1$, $WF^n = \{Y : Y \text{ is wellfounded on } A^n\}$.
By $\Delta_1^1(\phi_0)$-Comp we mean the class of all the sentences of L_2^{U} of the form

$$\forall \bar{Y} \; [\forall \bar{x} \; (\exists Z_1 \phi(\bar{x},Z_1,\bar{Y}) \leftrightarrow \forall Z_2 \Psi(\bar{x},Z_2,\bar{Y}))$$

$$\rightarrow \exists X \forall \bar{z} \; (\bar{z} \in X \leftrightarrow \exists Z_1 \phi(\bar{z},Z_1,\bar{Y}))]$$

where X does not occur in ϕ and $\phi, \Psi \in Ar(\phi_0)$.
By $\Sigma_1^1(\phi_0)$-Comp we mean the class of all the sentences of L_2^{U} of the form

$$\forall \bar{Y} \exists X \forall \bar{z} \; (\bar{z} \in X \leftrightarrow \phi(\bar{z},\bar{Y}))$$

where $\phi \in \Sigma_1^1(\phi_0)$ and X does not occur in ϕ.
It is clear what we mean by Δ_1^1-Comp, Σ_1^1-Comp or $\Sigma_1^1(\phi_0,WF(X))$-Comp.

LEMMA 2. *Let ϕ_0 be a formula of L_2^{U}. For every $\Sigma_1^1(\phi_0)$ formula Ψ of L_2^{U} there exists an $Ar(\phi_0)$ formula ϕ such that if A is second order structure over U which satisfies the following two conditions*

(1) $X,Y \in |A| \Rightarrow X \times Y \in |A|$

(2) *If $X \in |A|$ and X is a $\ell+m$-ary set ($\ell,m \geq 1$) then there exist sets $X_1, X_2 \in |A|$ ℓ-ary and m-ary respectively such that*

$$A \models \forall \bar{z} \; (\bar{z} \in X_1 \leftrightarrow \exists \bar{y} \; ((\bar{z},\bar{y}) \in X))$$

$$A \models \forall \bar{y} \; (\bar{y} \in X_2 \leftrightarrow \exists \bar{z} \; ((\bar{z},\bar{y}) \in X))$$

then

$$A \models \Psi \leftrightarrow \exists X \phi.$$

PROOF. Assume for simplicity that Ψ is of the form $\exists X_1 \exists X_2 \Psi_1(\bar{x}, X_1, X_2, \bar{X})$ where $\Psi_1(\bar{x}, X_1, X_2, \bar{X})$ is a $(k, \ell, m, n_1, \ldots, n_s)$ $Ar(\phi_0)$-formula of L_2^u for some $k, \ell, m, n_1, \ldots, n_s$ where $\ell, m \geq 1$.

Let $\Psi'(\bar{x}, X, X_2, \bar{X})$ be the following $(k, \ell+m, n_1, \ldots, n_s)$-formula of L_2^u

$$\Psi_1 \frac{}{z}(\bar{x}, \exists \bar{y}((\bar{z}, \bar{y}) \epsilon X), X_2, \bar{X})$$

and let $\phi(\bar{x}, X, \bar{X})$ be the $(k, \ell+m, n_1, \ldots, n_s)$-formula of L_2^u obtained from $\Psi' \frac{}{y}(\bar{x}, X, \exists \bar{z}((\bar{z}, \bar{y}) \epsilon Y, \bar{X})$ by replacing all the occurrences of Y by X (we have to make this small detour via Y in order to avoid the clash of variables). Clearly $\phi \epsilon Ar(\phi_0)$.

Now it is easy to see that for all $\bar{x}$ and $\bar{X}$

$$A \models \exists X_1 \exists X_2 \Psi_1(\bar{x}, X_1, X_2, \bar{X}) \leftrightarrow \exists X \phi(\bar{x}, X, \bar{X}).$$

Indeed, if for some X_1 and X_2 $A \models \Psi_1(\bar{x}, X_1, X_2, \bar{X})$ then $X_1 \times X_2 \epsilon |A|$ and clearly by lemma 1 $A \models \phi(\bar{x}, X, \bar{X})$.

Conversely, if for some X $A \models \phi(\bar{x}, X, \bar{X})$ then

$$X_1 = \{\bar{z} : A \models \exists \bar{y}((\bar{z}, \bar{y}) \epsilon X)\} \epsilon |A|$$

$$X_2 = \{\bar{y} : A \models \exists \bar{z}((\bar{z}, \bar{y}) \epsilon X)\} \epsilon |A|$$

and clearly by lemma 1

$$A \models \Psi_1(\bar{x}, X_1, X_2, \bar{X}). \quad \Box$$

COROLLARY 1. *Let ϕ_0 be a formula of L_2^u. Let A be a second order structure over U. If $A \models \Delta_1^1(\phi_0)$-Comp then A also satisfies the following scheme*

$$\forall \bar{Y} \; [\forall \bar{z}(\phi(\bar{z},\bar{Y}) \leftrightarrow \Psi(\bar{z},\bar{Y})) \rightarrow \exists X \forall \bar{z}(\bar{z}\epsilon X \leftrightarrow \phi(\bar{z},\bar{Y}))]$$

where X does not occur in ϕ *and* ϕ,Ψ *are* $\sum_1^1(\phi_0)$ *formulas.*

PROOF. By lemma 2   □

Suppose now that $\phi(x_1,\ldots,x_n,Y)$ is a (n,n)-formula of L_2^U. ϕ defines an operator Φ (not necessarily monotone) on the n-ary relations

$$\Phi(S) = \{(x_1,\ldots,x_n) \; : \; P(U) \models \phi(x_1,\ldots,x_n,S)\}$$

Define by induction

$$I_\phi^\xi = \bigcup_{\eta<\xi} I_\phi^\eta \cup \Phi \left(\bigcup_{\eta<\xi} I_\phi^\eta \right)$$

$$I_\phi = \bigcup_\xi I_\phi^\xi$$

By the closure ordinal of $\phi, \|\phi\|$, we mean the least ξ such that $I_\phi^\xi = \bigcup_{\eta<\xi} I_\phi^\eta$. I_ϕ is said to be inductively defined by ϕ. Observe that

$$I_\phi = I_\phi^{\|\phi\|} = \bigcup_{\eta<\|\phi\|} I_\phi^\eta.$$

By $|X|$ we denote the cardinality of a set X. By η^+ we mean the least cardinal number greater than η.

If ϕ_0 is a (n,n)-formula of L_2^U where $U = <A,R_1,\ldots,R_n>$ then $\|\phi_0\| < |A|^+$. If $\bar{x} \in A^n$ then

$$|\bar{x}|_{\phi_0} = \begin{cases} \text{least } \eta \text{ such that } \bar{x} \in I_{\phi_0}^\eta & \text{if } \bar{x} \in I_{\phi_0} \\ |A|^+ & \text{otherwise.} \end{cases}$$

Observe that for every $\eta < |A|^+$ there exists $X \subset A^2$ such that $P(U) \models WF(X)$ and $\|X\| = \eta$.

§3. NONMONOTONE INDUCTIVE DEFINITIONS AND MODELS OF COMPREHENSION

The first theorem we prove is the following:

THEOREM 1. Let ϕ_0 be a $(1,1)$-formula of L_2^U. Let A be a second order struc-
ture over U such that $A \models \Delta_1^1(\phi_0)$-Comp and ϕ_0 is A-absolute. Then

$$\eta < h(A) \Rightarrow I_{\phi_0}^\eta \in |A|.$$

PROOF. Define a $(2,2,2)$-formula $\Phi(x,y,X,Y)$ of L_2^U as follows:

$$\Phi(x,y,X,Y) \leftrightarrow \exists z\ (z\in Fld(X\restriction_x) \land (z,y)\in Y)$$

Let

$$\Psi(x,y,X,Y) \leftrightarrow \Phi(x,y,X,Y) \lor [\phi_{0y'}(y,\Phi(x,y',X,Y) \land x \in Fld(X)]$$

(we avoid the clash of variables by appropriate renaming of the variables
occurring in ϕ_0).
   Finally define

$$\theta(X,Y) \leftrightarrow \forall x\forall y((x,y) \in Y \leftrightarrow \Psi(x,y,X,Y))$$

Observe that

$$\Psi(x,y,X,Y) \in Ar(\phi_0)$$

and

$$\theta(X,Y) \in Ar(\phi_0)$$

so both are A-absolute formulas.
   We prove at first the following two facts:

I) If $P(\mathcal{U}) \models WF(X)$ then for all Y

$$P(\mathcal{U}) \models \theta(X,Y) \leftrightarrow Y = \left\{(x,y) : x \in Fld(X), \ y \in I_{\phi_0}^{\|X\restriction_x\|}\right\}$$

II) If $X \in |A|$ and $P(\mathcal{U}) \models WF(X)$ then

$$\{(x,y) : x \in Fld(X), \ y \in I_{\phi_0}^{\|X\restriction_x\|}\} \in |A|$$

Suppose that $P(\mathcal{U}) \models WF(X)$. Let

$$Y_0 = \{(x,y) : x \in Fld(X), \ y \in I_{\phi_0}^{\|X\restriction_x\|}\}$$

PROOF OF I). We prove that $P(\mathcal{U}) \models \theta(X,Y_0)$.

Let x and y be arbitrary. If $x \notin Fld(X)$ then

$$P(\mathcal{U}) \models (x,y) \notin Y_0 \land \neg\Psi(x,y,X,Y_0).$$

So assume that $x \in Fld(X)$. Then

$$P(\mathcal{U}) \models (x,y) \in Y_0 \leftrightarrow y \in I_{\phi_0}^{\|X\restriction_x\|}$$

$$\leftrightarrow y \in \bigcup_{\eta < \|X\restriction_x\|} I_\phi^\eta 0 \lor \phi_0\left(y, \bigcup_{\eta < \|X\restriction_x\|} I_\phi^\eta 0\right)$$

because $(z,x) \in X \Rightarrow (X\restriction_x \restriction_z = X\restriction_z) \leftrightarrow \exists z\left(z\in Fld(X\restriction_x) \land y \in I_{\phi_0}^{\|X\restriction_z\|}\right) \lor$

$$\phi_0\left(y, \left\{y' : P(\mathcal{U}) \models \exists z(z\in Fld(X\restriction_x) \land y' \in I_{\phi_0}^{\|X\restriction_z\|}\right\}\right)$$

by lemma 1 $\qquad \leftrightarrow P(\mathcal{U}) \models \Phi(x,y,X,Y_0) \lor \phi_{0y'}(y,\Phi(x,y',X,Y_0))$

$$\leftrightarrow P(\mathcal{U}) \models \Psi(x,y,X,Y_0)$$

Thus $P(U) \models \theta(X,Y_0)$.

Suppose now that for some Y $P(U) \models \theta(X,Y)$.
Observe that then for all x and y

$$(x,y) \in Y \rightarrow x \in Fld(X).$$

We prove by induction with respect to $\|X\!\upharpoonright_x\|$ that for all x,y

$$(x,y) \in Y \leftrightarrow (x,y) \in Y_0.$$

Let $x \in Fld(X)$. Suppose that the claim is true for all pairs (z,y) such that $\|X\!\upharpoonright_z\| < \|X\!\upharpoonright_x\|$.
Then for all y

$$P(U) \models (x,y) \in Y \leftrightarrow P(U) \models \Psi(x,y,X,Y)$$

by lemma 1

$$\leftrightarrow P(U) \models \exists z(z \in Fld(X\!\upharpoonright_x) \wedge (z,y) \in Y)$$

$$\vee \phi_0\left(y, \{y' : P(U) \models \exists z(z \in Fld(X\!\upharpoonright_x) \wedge (z,y) \in Y)\}\right)$$

By induction hypothesis

$$\leftrightarrow P(U) \models \exists z(z \in Fld(X\!\upharpoonright_x) \wedge (z,y) \in Y_0)$$

$$\vee \phi_0\left(y, \{y' : P(U) \models \exists z(z \in Fld(X\!\upharpoonright_x) \wedge (z,y) \in Y_0)\}\right)$$

by lemma 1

$$\leftrightarrow P(U) \models \Psi(x,y,X,Y_0)$$

by the above string
of equivalences

$$\leftrightarrow P(U) \models (x,y) \in Y_0.$$

Thus by induction $Y = Y_0$ which concludes the proof of I).

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PROOF OF II). Assume additionally that $X \in |A|$. We prove the claim by induction with respect to $\|X\|$. So suppose that the claim is true for $Z \in |A|$ such that $\|Z\| < \|X\|$. Let for $x \in Fld(X)$

$$Y(x) = \left\{ (z,y) \; : \; z \in Fld(X\restriction_x), \; y \in I_{\phi_0}^{\|X\restriction_x\restriction_z\|} \right\}$$

For $x \in Fld(X)$ $\|X\restriction_x\| < \|X\|$ and $X\restriction_x \in |A|$ so by induction hypothesis and I) combined with A-absoluteness of θ

(3) $\forall x \left[x \in Fld(X) \to \Big(Y(x) \in |A| \text{ and } A \models \theta(X\restriction_x, Y(x)) \Big) \right]$

We prove now that

(4) $A \models \forall x \forall y \left[\exists T \exists Y (x \in Fld(X) \land X\restriction_x = T \land \theta(T,Y) \land \Psi(x,y,X,Y)) \right.$

$\qquad\qquad \left. \leftrightarrow \forall T \forall Y \Big(x \in Fld(X) \land ((X\restriction_x = T \land \theta(T,Y)) \to \Psi(x,y,X,Y)) \Big) \right]$

Take arbitrary x and y. Suppose that the left hand side of the equivalence holds. Then $x \in Fld(X)$. By A-absoluteness of θ, I) and (3)

$$A \models \Psi(x,y,X,Y(x)).$$

Take now arbitrary Y such that $A \models \theta(X\restriction_x, Y)$. Then $P(U) \models \theta(X\restriction_x, Y)$, so by I) $Y = Y(x)$. By the above $A \models \Psi(x,y,X,Y)$. Thus $A \models \forall Y(\theta(X\restriction_x, Y) \to \Psi(x,y,X,Y))$ i.e. the right hand side of the equivalence holds.

Conversely, suppose that the right hand side holds. Then $x \in Fld(X)$. By (3) $Y(x) \in |A|$ and $A \models \theta(X\restriction_x, Y(x))$. Thus $A \models \Psi(x,y,X,Y(x))$. Hence $A \models \exists Y(\theta(X\restriction_x, Y) \land \Psi(x,y,X,Y))$ i.e. the left hand side holds.

So (4) is proved. A satisfies $\Delta_1^1(\phi_0)$-Comp, so by (4) and corollary 1

$$A \models \forall x \forall y \left[(x,y) \in Y_1 \leftrightarrow \exists T \exists Y (x \in Fld(X) \land X\restriction_x = T \land \theta(T,Y) \land \Psi(x,y,X,Y)) \right]$$

for some $Y_1 \in |A|$.

for all x and y

$$A \models (x,y) \in Y_1 \Leftrightarrow A \models x \in Fld(X) \wedge \exists Y(\theta(X\restriction_x, Y) \wedge \Psi(x,y,X,Y))$$

$) and (3) $\qquad \Leftrightarrow A \models x \in Fld(X) \wedge \Psi(x,y,X,Y(x))$

ə Ψ is A-absolute $\quad \Leftrightarrow P(\mathcal{U}) \models x \in Fld(X) \wedge \Psi(x,y,X,Y(x))$

ɪe definition of $\Psi \quad \Leftrightarrow x \in Fld(X) \ \& \ P(\mathcal{U}) \models \exists z\Big(z \in Fld(X\restriction_x) \wedge y \in I_{\phi_0}^{\|X\restriction_x\restriction_z\|}\Big)$

lemma 1 $\qquad \vee \ \phi_0\Big(y, \big\{ y' : P(\mathcal{U}) \models \exists z (z \in Fld(X\restriction_x) \wedge y' \in I_{\phi_0}^{\|X\restriction_x\restriction_z\|}) \big\}\Big)$

ɪe definition of $\|X\restriction_x\| \quad \Leftrightarrow x \in Fld(X) \ \& \ P(\mathcal{U}) \models y \in \bigcup_{\eta < \|X\restriction_x\|} I_{\phi_0}^{\eta}$

$$\vee \ \phi_0\Big(y, \bigcup_{\eta < \|X\restriction_x\|} I_{\phi_0}^{\eta}\Big)$$

$$\cdot \qquad \Leftrightarrow x \in Fld(X) \ \& \ y \in I_{\phi_0}^{\|X\restriction_x\|}.$$

$Y_1 = Y_0$. But $Y_1 \in |A|$ so the proof of II) is concluded.

Let now $\eta < h(A)$. Then for some $X \in |A|$ such that $P(\mathcal{U}) \models WF(X)$
$X\|$. By II)

$$Y_0 = \Big\{ (x,y) : x \in Fld(X), \ y \in I_{\phi_0}^{\|X\restriction_x\|} \Big\} \in |A|$$

ɪmprehension there exists $Z \in |A|$ such that

$$A \models \forall y \Big(y \in Z \leftrightarrow \exists x((x,y) \in Y_0) \vee \phi_{0y'}(y', \exists x((x,y') \in Y_0))\Big)$$

Observe that for all y

$$A \models y \in Z \qquad \leftrightarrow P(U) \models \exists x((x,y) \in Y_0) \vee \phi_{0y'}\Big(y, \exists x((x,y') \in Y_0))\Big)$$

by lemma 1
$$\leftrightarrow y \in \bigcup_{\xi < \|X\|} I^{\eta}_{\phi_0} \vee P(U) \models \phi_0\Big(y, \bigcup_{\xi < \|X\|} I^{\eta}_{\phi_0}\Big).$$

$$\leftrightarrow y \in I^{\|X\|}_{\phi_0}$$

Thus $Z = I^{\eta}_{\phi_0}$ which concludes the proof. $\square$

What are the conditions which imposed on a structure A imply that $\|\phi_0\| < h(A)$ and $I_{\phi_0} \in |A|$? Clearly we have to assume that A is a β-structure — there are ω-models of full-comprehension which do not contain $\Pi^1_1 - \Sigma^1_1$ sets (see e.g. APT [1]), so in particular do not contain all sets inductively defined by arithmetical formulas.

Is it sufficient to assume that A is a β-model of $\Delta^1_1(\phi_0)$-Comp? The answer is no, because there exist acceptable structures on which every model of Δ^1_1-Comp is a β-model.

We conjecture that $\Sigma^1_1(\phi_0)$-Comprehension is not sufficient either. Clearly one has to add the condition that ϕ_0 is A-absolute.

We prove the following theorem:

THEOREM 2. Let ϕ_0 be a $(1,1)$-formula of L^U_2. Suppose that A is a β-structure over U which is a model of $\Sigma^1_1(\phi_0, WF(X))$-Comp and that ϕ_0 is A-absolute. Then $\|\phi_0\| < h(A)$.

PROOF. Let

$$X(x,y) \leftrightarrow \exists X \exists Y [WF(X) \wedge \theta(X,Y) \wedge \exists z((z,x) \in Y \wedge (z,y) \notin Y)]$$

where $\theta(X,Y)$ is the formula defined in the proof of theorem 1. $X(x,y)$ is a $\Sigma^1_1(\phi_0, WF(X))$-formula, so there exists $Z \in |A|$ such that for all x and y

$$A \models (x,y) \in Z \leftrightarrow X(x,y).$$

Observe that for all x and y

$$(x,y) \in Z \Leftrightarrow A \models WF(X) \wedge \theta(X,Y) \wedge \exists z((z,x) \in Y \wedge (z,y) \notin Y)$$

for some $X \in |A|$ and $Y \in |A|$

because A is a β-structure $\Leftrightarrow P(U) \models \theta(X,Y) \wedge \exists z((z,x) \in Y \wedge (z,y) \notin Y)$
and $\theta(X,Y)$ is A-absolute

for some $X \in |A|$ and $Y \in |A|$ such that $P(U) \models WF(X)$

by I) and II)
$$\Leftrightarrow \exists z\left(z \in Fld(X) \wedge x \in I_{\phi_0}^{\|X\upharpoonright_z\|} \wedge y \notin I_{\phi_0}^{\|X\upharpoonright_z\|}\right)$$

for some $X \in |A|$ such that $P(U) \models WF(X)$

because $h(A)$ is limit
$$\Leftrightarrow x \in I_{\phi_0}^{\eta} \wedge y \notin I_{\phi_0}^{\eta} \text{ for some } \eta < h(A)$$

$$\Leftrightarrow |x|_{\phi_0} < h(A) \ \& \ |x|_{\phi_0} < |y|_{\phi_0}.$$

It follows that Z is transitive and wellfounded i.e. $P(U) \models WF(Z)$.

Suppose that $h(A) \leq \|\phi_0\|$. Then for every $\eta < h(A)$ there exists $x \in Fld(Z)$ such that $|x|_{\phi_0} = \eta$. Clearly $\|Z\upharpoonright_x\| = \eta$. Thus $h(A) \leq \|Z\|$, which is a contradiction. $\square$

THEOREM 3. Let ϕ_0 be a $(1,1)$-formula of L_2^U, where $U = <A,R_1,\ldots,R_n>$. Let

$$K(\phi_0) = \{A : A \text{ is a } \beta\text{-structure over } U, \quad A \models \Sigma_1^1(\phi_0, WF(X))\text{-Comp}$$
$$\text{and } \phi_0 \text{ is } A\text{-absolute}\}$$

Then

i) $I_{\phi_0} \in \cap K(\phi_0)$.

ii) $I_{\phi_0} \in Def(K(\phi_0))$.

Moreover I_{ϕ_0} is invariantly definable over $K(\phi_0)$ by a $\Sigma_1^1(\phi_0, WF(X))$-formula.

iii) There exists a $\sum_1^1(\phi_0, WF(X))$-formula $\Psi_0(x,y)$ such that

    i') $\Psi_0(x,y)$ is A-absolute for $A \in K(\phi_0)$

    ii') $\|Z_0\| = \|\phi_0\|$ where $Z_0 = \{(x,y) : P(U) \models \Psi_0(x,y)\}$

PROOF OF i). By theorem 1 and 2.

PROOF OF ii). Observe that for all x

$$P(U) \models x \in I_{\phi_0} \leftrightarrow \exists X \exists Y [WF(X) \wedge \theta(X,Y) \wedge \exists z((z,x) \in Y)]$$

Indeed,

$$x \in I_{\phi_0} \leftrightarrow x \in I_{\phi_0}^{\eta+1} \text{ for some } \eta < |A|^+$$

$$\leftrightarrow x \in I_{\phi_0}^{\|X \upharpoonright z\|} \text{ for some X such that WF(X)}$$

$$\text{and } z \in Fld(X)$$

by I) $\leftrightarrow \exists X \exists Y [WF(X) \wedge \theta(X,Y) \wedge \exists z((z,x) \in Y)].$

It is left to prove that the formula

$$\exists X \exists Y [WF(X) \wedge \theta(X,Y) \wedge \exists z((z,x) \in Y)]$$

is A-absolute for $A \in K(\phi_0)$.

Let $A \in K(\phi_0)$. By theorem 2 there exists $Z \in |A|$ such that $P(U) \models WF(Z)$ and $\|Z\| = \|\phi_0\|$.

We have for all x

$$P(U) \models \exists X \exists Y [WF(X) \wedge \theta(X,Y) \wedge \exists z((z,x) \in Y)]$$

$$\Rightarrow x \in I_{\phi_0}$$

$$\Rightarrow x \in I_{\phi_0}^{\eta} \text{ for some } \eta < \|Z\|$$

$$\Rightarrow x \in I_{\phi_0}^{\|Z \upharpoonright z\|} \text{ for some } z \in Fld(Z)$$

by I) $\Rightarrow P(U) \models \exists Y [\theta(Z,Y) \wedge \exists z((z,x)\epsilon Y)]$

by II) $\Rightarrow A \models \exists Y [\theta(Z,Y) \wedge \exists z((z,x)\epsilon Y)]$

$\Rightarrow A \models \exists X \exists Y [WF(X) \wedge \theta(X,Y) \wedge \exists z((z,x)\epsilon Y)]$

Implication in the other direction follows from the fact that $WF(X)$ and $\theta(X,Y)$ are A-absolute formulas.

PROOF OF iii). Let

$$\Psi_0(x,y) \leftrightarrow \exists X \exists Y [WF(X) \wedge \theta(X,Y) \wedge \exists t((t,y)\epsilon Y) \wedge \exists z((z,x) \epsilon Y \wedge (z,y)\notin Y)]$$

and let $Z_0 = \{(x,y) : P(U) \models \Psi_0(x,y)\}$.
Then for all x and y

$$P(U) \models (x,y) \epsilon Z_0 \leftrightarrow y \epsilon I_{\phi_0} \text{ \& } |x|_{\phi_0} < |y|_{\phi_0}$$

as a string of equivalences analogous to that in the proof of theorem 2 shows (in place of A we should take $P(U)$).

The proof that $\Psi_0(x,y)$ is A-absolute for $A \epsilon K(\phi_0)$ is analogous to the appropriate proof from part ii). $\square$

Let $WF^n(X)$ be the following $(0,2n)$-formula of L_2^U where $U = <A,R_1,\ldots,R_n>$;
$WF^n(X) \leftrightarrow X$ is a wellfounded transitive relation on A^n.

Clearly all the results of this section hold for (n,n)-formulas instead of $(1,1)$-ones after replacing throughout the section each occurrence of the formula $WF(X)$ (including the definition of $h(A)$ - we denote this changed ordinal by $h^n(A)$) by $WF^n(X)$.

From the proofs of theorems 1 and 2 we can easily extract

THEOREM 4. Suppose that for some $n \geq 1$ the relation WF^n is arithmetical on U.

Then

$$\{I_{\phi_0} : \phi_0 \text{ is a } (n,n)\text{-arithmetical formula of } L_2^{\mathcal{U}}\} \subseteq \cap Mod(\textstyle\sum_1^1\text{-Comp})$$

PROOF. For some arithmetical formula Ψ of $L_2^{\mathcal{U}}$

$$P(\mathcal{U}) \models WF^n(X) \leftrightarrow \Psi(X)$$

Suppose that ϕ_0 is a (n,n)-arithmetical formula of $L_2^{\mathcal{U}}$. Then $\Delta_1^1(\phi_0)\text{-Comp} = \Delta_1^1\text{-Comp}$. Let A be a second order structure over $\mathcal{U}$ which is a model of $\textstyle\sum_1^1\text{-Comp}$.

Let $X(\bar{x},\bar{y})$ be the following $(2n)$-formula of $L_2^{\mathcal{U}}$

$$X(\bar{x},\bar{y}) \leftrightarrow \exists X \exists Y [\Psi(X) \wedge \theta(X,Y) \wedge \exists \bar{z}((\bar{z},\bar{x}) \in Y \wedge (\bar{z},\bar{y}) \notin Y)]$$

where $\theta(X,Y)$ is the appropriate $(0,2n,2n)$-formula defined in the proof of theorem 1.

$X(\bar{x},\bar{y})$ is a $\textstyle\sum_1^1$ formula, so there exists $Z \in |A|$ such that

$$A \models \forall \bar{x} \forall \bar{y}((\bar{x},\bar{y}) \in Z \leftrightarrow X(\bar{x},\bar{y})).$$

The same string of equivalences like in the proof of theorem 2 shows that for all $\bar{x}$ and $\bar{y}$

$$(\bar{x},\bar{y}) \in Z \leftrightarrow |\bar{x}|_{\phi_0} < h^n(A) \ \& \ |\bar{x}|_{\phi_0} < |\bar{y}|_{\phi_0}$$

which analogously implies that $\|\phi_0\| < h(A)$. Due to theorem 1 $I_{\phi_0} \in |A|$. $\square$

COROLLARY 2. *Barwise-Grilliot theorem (see MOSCHOVAKIS [2] p. 140) is false if one omits the assumption of countability.*

PROOF. There are acceptable structures $\mathcal{U}$ on which all the relations WF^n are arithmetical (see MOSCHOVAKIS [2] theorem 6C.3). By theorem 4 the intersection of all models of $\textstyle\sum_1^1\text{-Comp}$ contains all inductive relations on $\mathcal{U}$, so by corollary 5D.3 from MOSCHOVAKIS [2] it is different from the family of all hyperarithmetical relations on $\mathcal{U}$. $\square$

NOTE. The other way to prove the above corollary is to use theorem 7A.1 form MOSCHOVAKIS [2] in connection with theorem 7F.1. The advantage of the proof presented above is that it is much more elementary – it doesn't even use any of two Stage Comparison Theorems.

§4. INVARIANT DEFINABILITY OVER MODELS OF Δ^1_1-COMPREHENSION.

Throughout this section we shall use freely several results from MOSCHOVAKIS [2]. Instead of writing e.g. "by theorem 7A.2 from MOSCHOVAKIS [2]" we shall simply write "by theorem 7A.2".
All the previously unexplained notations can be found in MOSCHOVAKIS [2].

THEOREM 5. Let U be an acceptable structure. Let T be an inductive theory in L_2^U which has a second order model over U. Then

$$\text{Def}(\text{Mod}(T)) \subseteq \Delta^1_1.$$

Here, of course, Δ^1_1 denotes the family of all Δ^1_1 first order relations on U.

PROOF. Let $X \in \text{Def}(\text{Mod }(T))$. For some formula ϕ of L_2^U which is A-absolute for $A \in \text{Mod}(T)$

$$P(U) \models \forall \bar{x}(\bar{x} \in X \leftrightarrow \phi(\bar{x})).$$

Thus for all $\bar{x} = (x_1, \ldots, x_n)$

$$\bar{x} \in X \leftrightarrow P(U) \models \phi(\bar{x})$$

$$\leftrightarrow \text{for every } A \in \text{Mod}(T) \quad A \models \phi(\bar{x})$$

$$\leftrightarrow \forall Z \, [\text{Mod}^T(Z) \to \text{Sat}_2(\ulcorner \phi \urcorner, Z, \phi, <x_1, \ldots, x_n>)]$$

(we are using here the notation of lemma 7E.2 and theorem 8C.2). $\text{Mod}^T(Z)$ is a Σ^1_1 relation (see theorem 8C.2) and $\text{Sat}_2(a,Z,Y,b)$ is a Δ^1_1 relation (see lemma 7E.2 and theorem 6B.5). Thus X is a Π^1_1 relation. Also $\neg X \in \text{Def}(\text{Mod}(T))$, so X is a Π^1_1 relation as well. Thus X is Δ^1_1. ⬜

THEOREM 6. Let U be an acceptable structure. Let T be a theory in L_2^U such that Δ_1^1-Comp $\subset$ T and T has a second order model over U.
Then

$$\mathcal{HK}(U) \subset \text{Def}(\text{Mod}(T)).$$

PROOF. Suppose that $X_0 \in \mathcal{HK}(U)$. By theorem 7D.1 there exists a formula $\phi(\bar{x},X)$ of L^U such that for all $\bar{x}$

$$\bar{x} \in X_0 \leftrightarrow \exists X \phi(\bar{x},X) \leftrightarrow \exists X (X \in \mathcal{HK}(U) \ \& \ \phi(\bar{x},X)).$$

Let $A \in \text{Mod}(T)$. Then $\mathcal{HK}(U) \subset |A|$ by theorem 7E.1.
We have for all $\bar{x}$

$$P(U) \models \exists X \phi(\bar{x},X) \Rightarrow P(U) \models \phi(\bar{x},X) \text{ for some } X \in \mathcal{HK}(U)$$

$$\Rightarrow P(U) \models \phi(\bar{x},X) \text{ for some } X \in |A|$$

$$\Rightarrow A \models \exists X \phi(\bar{x},X).$$

Thus $\exists X \phi(\bar{x},X)$ is A-absolute which shows that $X_0 \in \text{Def}(\text{Mod}(T))$. $\square$
Combining these two theorems we get

COROLLARY 3. *Let U be a countable acceptable structure. Let* T *be an inductive theory in* L_2^U *such that* Δ_1^1-Comp $\subset$ T *and* T *has a second order model over* U. *Then*

$$\mathcal{HK}(U) = \text{Def}(\text{Mod}(T)).$$

PROOF. By theorem 8A.1 $\mathcal{HK}(U) = \Delta_1^1$, so the claim follows from theorems 5 and 6.

The above statement is not true for uncountable acceptable structures. It follows from the following theorem.

THEOREM 7. Let U be an acceptable structure. Suppose that for some $n \geq 1$ WF^n is arithmetical on U. Then

$$\left\{ I_{\phi_0} : \phi_0 \text{ is a } (n,n) \text{ } X\text{-positive formula of } L^U \right\} \subset \text{Def}(\text{Mod}(\Delta_1^1\text{-Comp})).$$

PROOF. Let ϕ_0 be a (n,n) X-positive formula of L^U and let A be a model of Δ_1^1-Comp.

By theorem 7A.1 there exists a formula $\phi(\bar{x},Y)$ of L^U such that for all $\bar{x}$

$$\bar{x} \in I_{\phi_0} \leftrightarrow \exists Y \; [WF^n(Y) \; \& \; \phi(\bar{x},Y)]$$

$$\leftrightarrow \exists Y \; [Y \in \mathcal{HC}(U) \; \& \; WF^n(Y) \; \& \; \phi(\bar{x},Y)].$$

For some formula Ψ of L^U

$$WF^n(X) \leftrightarrow P(U) \models \Psi(X)$$

for all $X \subset A^{2n}$.
Then

$$P(U) \models \forall \bar{x}\left(\bar{x} \in I_{\phi_0} \leftrightarrow \exists Y \; [\Psi(Y) \wedge \phi(\bar{x},Y)] \right).$$

We have for all $\bar{x}$

$$P(U) \models \exists Y \; [\Psi(Y) \wedge \phi(\bar{x},Y)] \Rightarrow P(U) \models \Psi(Y) \wedge \phi(\bar{x},Y)$$

$$\text{for some } Y \in \mathcal{HC}(U)$$

by theorem 7F.1 $\Rightarrow P(U) \models \Psi(Y) \wedge \phi(\bar{x},Y)$

$$\text{for some } Y \in |A|$$

$$\Rightarrow A \models \exists Y \; [\Psi(Y) \wedge \phi(\bar{x},Y)]$$

which shows that $\exists Y \, [\Psi(Y) \wedge \theta(Y,\bar{x})]$ is A-absolute.

Thus $I_{\phi_0} \in \text{Def} \, (\text{Mod}(\Delta_1^1\text{-Comp}))$. $\square$

We are unable to characterize the sets which belong to $\text{Def}(\text{Mod}(\Delta_1^1\text{-Comp}))$ in the case of arbitrary acceptable structures. Observe, however that the following theorem holds.

THEOREM 8. Let $\mathcal{U} = \; < A, R_1, \ldots, R_n >$ be an acceptable structure. There exists an inductive set I such that

$$\text{Def}(\text{Mod}(\Delta_1^1\text{-Comp})) \subset \mathcal{HE}(\mathcal{U},I).$$

PROOF. Let I be an inductive binary relation which parametizes the unary inductive relations (see theorem 5D.2).

Let $X \in \text{Def}(\text{Mod}(\Delta_1^1\text{-Comp}))$. Then for all $x_1, \ldots, x_r$

$$P(\mathcal{U}) \models (x_1, \ldots, x_r) \in X \leftrightarrow \; < A, \mathcal{HE}(\mathcal{U}), R_1, \ldots, R_n > \; \models \phi(x_1, \ldots, x_r).$$

Let k be the highest arity of second order variables occurring in ϕ. Then clearly for all $x_1, \ldots, x_r$

$$P(\mathcal{U}) \models (x_1, \ldots, x_r) \in X \leftrightarrow \; < A, \bigcup_{m \leq k} \mathcal{HE}^m(\mathcal{U}), R_1, \ldots, R_n > \; \models \psi(x_1 \ldots, x_r).$$

By theorem 5D.4 for every ℓ there exists a $(\ell+1)$-ary inductive set I^ℓ which parametizes $\mathcal{HE}^\ell(\mathcal{U})$.

Let $J = \{< \underline{m}, a > \; : \; a \in I^m, m \leq k\}$

(here $\underline{m}$ is the m-th integer in the copy of ω in a coding scheme fixed throughout the proof).

Clearly J is inductive.

By lemma 7E.2 for all $x_1, \ldots, x_r$

$$P(\mathcal{U}) \models (x_1, \ldots, x_r) \in X \leftrightarrow \text{Sat}_2(\ulcorner \phi \urcorner, J, \phi, < x_1, \ldots, x_r >)$$

where Sat_2 is a second order hyperelementary relation.

This easily implies that X is hyperelementary on (U,J).

For some a $J = \{x : (a,x) \in I\}$ thus $X \in \mathcal{HE}(U,I)$. $\square$

It seems likely that the inclusion in the above theorem could be replaced by equality. We are, however inable to prove it.

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